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# Behavior of Cesaro Means of Energy Components for Non-Simple Thermoelastic Bodies

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The mixed initial boundary value problem in the context of non-simple thermoelastic materials is approached. The energy components of the solution to this problem are associated with so called Cesaro means. The main result of this study consists in demonstrating that Cesaro means of the kinetic and strain energies become asymptotically equal as time tends to infinity.

**Keywords:** Cesaro Mean, Non-Simple Materials, Equipartition, Kinetic Energy, Strain Energy.

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## 1. INTRODUCTION

Even classical elasticity does not consider the inner structure, the material response of materials to stimuli depends in a relevant way on its internal structure. Thus, it has been needed to develop some new mathematical models for continuum materials where this kind of effects was taken into account. Some of them are non-simple elastic solids. It is known that from a mathematical point of view, these materials are characterized by the inclusion of higher order gradients of displacement in the basic postulates.

The theory of non-simple elastic materials was first proposed by Toupin in his famous article.[1] Also among the first studies devoted to this material must mention those belonging to Green and Rivlin[2] and Mindlin.[3]

The interest to introduce high order derivatives consists in the fact that the possible configurations of the materials are clarified more and more finely by the values of the successive higher gradients.

As it is known, the constitutive equations of non-simple elastic solids are known to contain first and second order gradients, both contributing to dissipation. It is then interesting to understand the relevance of the two different dissipation mechanisms which can appear in the theory. In fact, the simultaneous presence of both mechanisms can be analyzed as well, with inessential changes in the proofs: In that situation, the behavior turns out to be the same as if only the high order dissipation appears in the equations.

In the last decade many studies have been devoted to non-simple materials. I remember only three of them, though differ on issues addressed, though in essence, they are dedicated to non simple materials. So, in the paper of Pata and Quintanilla,[4] the theory is linearized, and a uniqueness result is presented.

Also, the study of Martinez and Quintanilla[5] is devoted to study the incremental problem in the thermoelastic theory of non-simple elastic materials.

A theory of thermoelasticity for non-simple materials is derived within the framework of extended thermodynamics in the paper of Ciarletta.[6] The theory is linearized and a uniqueness result is presented. A Galerkin type solution of the field equations and fundamental solutions for steady vibrations are also studied. Other results regarding generalized thermoelastic materials can be found in the papers.[7–16] We will study the asymptotic partition of total energy for the solutions of the mixed initial boundary value problem defined within the context of non-simple thermoelastic materials.

The notion of asymptotic equipartition of energy components of the solution has appeared for the first time in the context of some abstract differential equations. This means that the kinetic and potential energy of a classical solution with finite energy become asymptotically equal in means as time tends to infinity. In studies that followed, this property is presented for physical systems governed by nondissipative hyperbolic partial differential equations or systems of such equations.

It is important to emphasize that the system of equations governing our mixed initial boundary value problem consists of hyperbolic equations with dissipation and, therefore, does not belong to one of the categories considered

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previously in literature of the subject. Furthermore, it is interesting to note that just by using the dissipation mechanism of the system, we can prove that equipartition occurs between the mean kinetic and strain energies. We follow these techniques not in an abstracted version of this question, but we prefer to emphasize the technique itself on the thermoelasticity of non-simple materials.

After we first write down the mixed initial boundary value problem defined in the above context, we shall establish some Lagrange type identities and, also, we introduce the Cesaro means of various parts of the total energy associated with the solutions. As a main result, based on these estimations, we establish in Section 3 the relations that describe the asymptotic behavior of the mean energies. The literature on this subject we find many papers which employ the various refinements of the Lagrange identity, as in the papers.[17–20] Also, the Cesaro means of energies to the solution of a mixed problem can be found in a lot of dedicated papers, as Day,[21] Levine[17] for instance.

## 2. BASIC EQUATIONS AND CONDITIONS

Consider that a bounded region $B$ of three-dimensional Euclidian space $R^3$ is occupied by a microstretch thermoelastic body, referred to the reference configuration and a fixed system of rectangular Cartesian axes. By $\bar{B}$ we denote the closure of $B$ and call $\partial B$ the boundary of the domain $B$. Let us assume that $\partial B$ is a piecewise smooth surface and designate by $n_i$ the components of the outward unit normal to the surface $\partial B$. Letters in boldface stand for vector fields. We use the notation $v_i$ to designate the components of the vector $v$ in the underlying rectangular Cartesian coordinate frame. Superposed dots stand for the material time derivative. We shall employ the usual summation and differentiation conventions: the subscripts are understood to range over integer (1, 2, 3). The summation over repeated subscripts is implied, and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

The spatial argument and time argument of a function will be omitted when there is no likelihood of confusion. We refer the motion of the body to a fixed system of rectangular Cartesian axes $Ox_i$, $i = 1, 2, 3$. Let us denote by $u_i$ the components of the displacement vector and by $\theta$ the temperature measured from the constant absolute temperature $\theta_0$ of the body in its reference state. As usual, we denote by $t_{ij}$ the components of the stress tensor.

By convention, we will address the theory and the notation in the way developed by Iesan in his book.[22] According to this, in the absence of body force and heat supply fields, the fields of basic equations for thermoelasticity of non-simple bodies are:

$$t_{ji,j} + \mu_{sji,sj} = \rho \ddot{u}_i \tag{1}$$

$$\rho T_0 \dot{\eta} = q_{i,i} \tag{2}$$

Where, Eq. (1) is the equation of motion and Eq. (2) is the equation of energy.

If we consider that the reference solid has a center of symmetry at each point, but is otherwise non-isotropic, then we have the following constitutive equations:

$$\begin{aligned} t_{ij} &= A_{ijrs}\varepsilon_{rs} + B_{ijpqr}\chi_{pqr} + a_{ij}(\theta + \alpha\dot{\theta}) \\ \mu_{ijk} &= B_{rsijk}\varepsilon_{rs} + C_{ijkmnr}\chi_{mnr} + c_{ijk}(\theta + \alpha\dot{\theta}) \\ \rho\eta &= -a_{ij}\varepsilon_{ij} - c_{ijk}\chi_{ijk} + a + d\theta + h\dot{\theta} - b_i\theta_i \\ q_i &= T_0[b_i\dot{\theta} + k_{ij}\theta_j] \end{aligned} \tag{3}$$

Through the following geometrical equations, we introduce the kinematic characteristics of the body

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i}, \quad \chi_{ijk} = u_{k,ij} \tag{4}$$

In the above equations we have used the following notations:
—$\rho$ is the reference constant mass density;
—$\theta_0$ is the constant absolute temperature of the body in its reference state;
—$u_i$ the components of the displacement vector;
—$\theta$ the temperature variation measured from the constant temperature $\theta_0$;
—$\varepsilon_{ij}$ and $\chi_{ijk}$ the components of the strain;
—$t_{ij}$ and $\mu_{ijk}$ the components of the hyper stress;
—$\eta$ is the specific entropy per unit mass;
—$q_i$ are the components of the heat flux vector.

Also, the coefficients $A_{ijrs}$, $B_{ijpqr}$, $C_{ijkmnr}$, $c_{ijk}$, $a_{ij}$, $k_{ij}$, $b_i$, $a$, $d$, $h$ and $\alpha$ are the characteristic constants of the material and they satisfy the following symmetry relations

$$\begin{aligned} &A_{ijrs} = A_{rsij} = A_{jirs}, \quad B_{ijpqr} = B_{jipqr} = B_{ijqpr} \\ &\text{mod} * 1.5\text{ cm}, \quad C_{ijkpqr} = C_{pqrijk} = C_{jikpqr} \\ &c_{ijk} = c_{jik}, \quad a_{ij} = a_{ji}, \quad k_{ij} = k_{ji} \end{aligned} \tag{5}$$

We wish to outline that the constants $a$, $d$ and $h$ are specific coefficients of the heat. The density $\rho$ and the temperature $\theta_0$ are given positive constants which satisfy the conditions

$$\rho > 0, \quad \theta_0 > 0 \tag{6}$$

From the entropy production inequality we obtain the following conditions

$$(d\alpha - h)\dot{\theta}^2 + 2b_i\dot{\theta}\theta_i + k_{ij}\theta_i\theta_j \geq 0 \tag{7}$$

and, according to the same entropy inequality, we assume that $A_{ijrs}$, $C_{ijkmnr}$ and $k_{ij}$ are positive tensors, i.e.,

$$\begin{aligned} &A_{ijrs}\xi_{ij}\xi_{rs} \geq k_0\xi_{ik}\xi_{ik}, k_0 > 0, \quad \forall \xi_{ik} = \xi_{ki} \\ &C_{ijkmnr}\zeta_{ijk}\zeta_{mnr} \geq k_1\zeta_{ijk}\zeta_{ijk}, \quad k_1 > 0, \quad \forall \zeta_{ijk} \end{aligned} \tag{8}$$

$$k_{ij}x_i x_j \geq k_2 x_i x_i, \quad k_2 > 0, \quad \forall x_i \tag{9}$$



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Taking into account the paper[23] of Green and Lindsay, we can assume that

$$a > 0, \quad h > 0, \quad d\alpha - h > 0 \tag{10}$$

To complete the system of field Eqs. (1)–(2) we add the following boundary conditions:

$$\begin{aligned} u_i = 0 \quad &\text{on } \partial B_1 \times [0, \infty) \\ t_{ji} n_j = 0 \quad &\text{on } \partial B_1^c \times [0, \infty) \\ \theta = 0 \quad &\text{on } \partial B_2 \times [0, \infty) \\ q_i n_i = 0 \quad &\text{on } \partial B_2^c \times [0, \infty) \end{aligned} \tag{11}$$

As usual, the notations $\partial B_1$ and $\partial B_2$ with respective complements $\partial B_1^c$ and $\partial B_2^c$ stand for the subsets of the surface $\partial B$ such that

$$\begin{aligned} \partial B_1 \cap \partial B_1^c = \partial B_2 \cap \partial B_2^c = \varnothing \\ \partial B_1 \cup \partial B_1^c = \partial B_2 \cup \partial B_2^c = \partial B \end{aligned} \tag{12}$$

Finally, the mixed initial boundary value problem is complete if we adjoin the following initial conditions

$$\begin{aligned} u_i(x,0) = u_i^0(x), \quad \dot{u}_i(x,0) = u_i^1(x) \\ \theta(x,0) = \theta^0(x), \quad \dot{\theta}(x,0) = \theta^1(x), \quad x \in B \end{aligned} \tag{13}$$

By introducing the constitutive Eq. (3), in Eqs. (1) and (2) we obtain the following system of equations

$$\rho \ddot{u}_i = A_{ijrs} u_{r,sj} + B_{ijpqr} u_{r,pqj} + a_{ij}(\theta_{,j} + \alpha \dot{\theta}_{,j}) \tag{14}$$

$$h\ddot{\theta} = -d\dot{\theta} + a_{ij}\dot{u}_{i,j} + c_{ijk}\dot{u}_{k,ij} + 2b_i\dot{\theta}_{,i} + k_{ij}\theta_{,ij} \tag{15}$$

equations hold for $(x,t) \in B \times (0, \infty)$.

Also, we can rewrite the boundary conditions by using the constitutive equations

$$\begin{aligned} u_i = 0 \quad &\text{on } \partial B_1 \times [0, \infty) \\ [A_{ijrs} u_{r,s} + B_{ijpqr} u_{r,pq} &+ a_{ij}(\theta + \alpha\dot{\theta})] \\ \times n_j = 0 \quad &\text{on } \partial B_1^c \times [0, \infty) \\ \theta = 0 \quad &\text{on } \partial B_2 \times [0, \infty) \\ k_{ij}\theta_{,j} n_i = 0 \quad &\text{on } \partial B_2^c \times [0, \infty) \end{aligned} \tag{16}$$

Naturally, a solution of the mixed initial boundary value problem of the theory of thermoelasticity of non-simple materials in the cylinder $\Omega_0 = B \times [0, \infty)$ is an ordered array $(u_i, \theta)$ which satisfies the system of Eqs. (14) and (15) for all $(x,t) \in \Omega_0$, the boundary conditions (16) and the initial conditions (13).

Let us observe that, in the case that $meas(\partial B_1) = 0$, there exists a family of rigid motions and null temperature which satisfies the Eqs. (14) and (15) and the boundary conditions (16). As a consequence, we can decompose the initial data $u_i^0$ and $u_i^1$ as follows

$$u_i^0 = u_i^* + U_i^0, \quad u_i^1 = \dot{u}_i^* + U_i^1 \tag{17}$$

where $u_i^*$ and $\dot{u}_i^*$ are determined from the condition that the functions $U_i^0$ and $U_i^1$ satisfy the so-called normalization restrictions ($\varepsilon_{ijk}$ is Ricci's tensor)

$$\begin{aligned} \int_B \rho U_i^0 \, dV = 0, \quad \int_B \rho \varepsilon_{ijk} x_j U_k^0 \, dV = 0 \\ \int_B \rho U_i^1 \, dV = 0, \quad \int_B \rho \varepsilon_{ijk} x_j U_k^1 \, dV = 0 \end{aligned} \tag{18}$$

In an analogous manner, in the case $meas(\partial B_2) = 0$, which is equivalent to $\partial B_2 = \partial B$, there exists a family of constant temperatures and null displacements, which satisfy the Eqs. (14) and (15) and the boundary conditions (16). For this reason, we can decompose the initial data $\theta^0$ and $\theta^1$ as follows

$$\theta^0 = \theta^* + T^0, \quad \theta^1 = \dot{\theta}^* + \dot{T}^0 \tag{19}$$

Constants $\theta^*$ and $\dot{\theta}^*$ from relations (19) are determined from restrictions

$$\int_B T^0 \, dV = 0, \quad \int_B T^1 \, dV = 0 \tag{20}$$

In what follows, will be useful notations that we introduce now.

First, we denote by $C^m(B)$ the class of scalar fields possessing derivatives up to the $m$-th order in the domain $B$ which are continuous on $B$.

For $f \in C^m(B)$ we define the norm

$$\|f\|_{C^m(B)} = \sum_{k=0}^{m} \sum_{i_1, i_2, \ldots, i_k} \max |f_{,i_1 i_2, \ldots, i_k}|$$

We denote by $\mathbf{C}^m(B)$ the class of vector fields with three components $C^m(B)$.

For $\mathbf{W} \in \mathbf{C}^m(B)$ we define the norm

$$\|\mathbf{W}\|_{\mathbf{C}^m(B)} = \sum_{i=1}^{3} \|\mathbf{W_i}\|_{C^m}(B)$$

By $W_m(B)$ we denote the Hilbert space obtained as the completion of the space $C^m(B)$ by means of the norm $\|\cdot\|_{W_m}(B)$ induced by the inner product

$$(f,g)_{W_m}(B) = \sum_{i=0}^{m} \int_B f_{,i_1 i_2, \ldots, i_k} g_{,i_1 i_2, \ldots, i_k} \, dV$$

Finally, we will denote by $\mathbf{W}_m(B)$ the space obtained as the completion of the space $\mathbf{C}^m(B)$ by means of the norm $\|\cdot\|_{\mathbf{W}_m}(B)$ induced by the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathbf{W}_m(B)} = \sum_{i=1}^{3} (u_i, v_i)_{W_m}(B)$$

We will use as a norm in Cartesian products of the normed spaces the sum of the norms of the factor spaces.

Now mention other notations that we will use in the following.

$$\hat{C}^1(B) = \left\{\chi \in C^1(B)\colon \chi = 0 \text{ on } \partial B_2;\right.$$
$$\left.\text{if } meas(\partial B_2) = 0 \text{ then } \int_B \chi\, dV = 0\right\},$$
$$\mathbf{C}^1(B) = \left\{(v_1, v_2, v_3) \in C^1(B)\colon v_i = 0 \text{ on } \partial B_1\right.$$
$$\text{if } meas(\partial B_1) = 0, \text{ then } \int_B \rho v_i\, dV = 0$$
$$\left.\int_B \rho\varepsilon_{ijk} x_j v_k^0\, dV = 0\right\}$$
$$\hat{W}_1(B) = \text{the completion of } \hat{C}^1(B)$$
$$\text{by means of the norm } \|\cdot\|_{\mathbf{W}_1}(B)$$
$$\mathbf{W}_1(B) = \text{the completion of } \mathbf{C}^1(B)$$
$$\text{by means of the norm } \|\cdot\|_{\mathbf{W}_1}(B)$$

In these notations $\mathbf{W}_m(B)$ represents the familiar Sobolev space (see Adams[24]) and $\mathbf{W}_m(B) = [W_m(B)]$.[3]

Based on the hypothesis (8), for all $\mathbf{v} \in \hat{\mathbf{W}}(B)$ the following Korn's inequality holds, (see Hlavacek, Necas,[25])

$$\int_B [A_{ijrs} v_{i,j} v_{r,s} + 2B_{ijpqr} v_{i,j} v_{r,pq} + C_{ijkpqr} v_{k,ij} v_{r,pq}]\, dV$$
$$\geq m_1 \int_B [v_i v_i + v_{i,j} v_{i,j} + v_{i,jk} v_{i,jk}]\, dv \tag{21}$$

where $m_1$ is a positive constant.

Also, based on the hypothesis (9), for all $\theta \in \hat{W}_1(B)$ the following Poincare's inequality holds

$$\int_B k_{ij}\theta_{,i}\theta_{,j}\, dV \geq m_2 \int_B k_{ij}\theta^2\, dV \tag{22}$$

where, $m_2$ is a positive constant.

In the case that $meas(\partial B_1) = 0$, then will be useful in the following to decompose the solution $(u_i, \theta)$ in the form

$$u_i = u_i^* + t\dot{u}_i^* + v_i, \quad \theta = \chi \tag{23}$$

in which the functions $((v_i), \chi) \in \mathbf{W}_1(B) \times \hat{W}_1(B)$ satisfy the system of Eqs. (14) and (15) with the boundary conditions (16) and the initial conditions

$$v_i(x,0) = U_i^0(x), \quad \dot{v}_i(x,0) = U_i^1(x)$$
$$\chi(x,0) = \theta^0(x), \quad \dot{\chi}(x,0) = \theta^1(x), \quad \text{on } B$$

If we are in the situation that $meas(\partial B_2) = 0$, then we shall use the relations (19) and (20) and the Eq. (2) in order to decompose the solution $(u_i, \theta)$ in the form

$$u_i = v_i, \quad \theta = \theta^* + \frac{h}{d}\left[1 - \exp\left(-\frac{d}{h}t\right)\right]\dot{\theta}^* + \chi \tag{24}$$

where the functions $((v_i), \chi) \in \mathbf{W}_1(B) \times \hat{W}_1(B)$ satisfy the system of Eqs. (14) and (15) with the boundary conditions (16) and the initial conditions

$$v_i(x,0) = u_i^0(x), \quad \dot{v}_i(x,0) = u_i^1(x),$$
$$\chi(x,0) = T^0(x), \quad \dot{\chi}(x,0) = T^1(x), \quad \text{on } B$$

## 3. PRELIMINARY RESULTS

In this section we shall prove some integral identities which are essential to demonstrate the relations that express the asymptotic partition of energy.

We will start with the first integral identity, in the following theorem, which, in fact, is a conservation law of total energy.

THEOREM 1. *Let* $((u_i), \theta)$ *be a solution of the mixed initial boundary value problem defined by* (14), (15), (16) *and* (13). *If we suppose that*

$$(u_i^0) \in \mathbf{W}_1(B), \quad (u_i^1) \in \mathbf{W}_0(B)$$
$$\theta^0 \in W_1(B), \quad \theta^1 \in W_0(B)$$

*then we have the following conservation law*

$$\begin{aligned} E(t) = \frac{1}{2}\int_B [&\rho\dot{u}_i(t)\dot{u}_i(t) + A_{ijrs}u_{r,s}(s)u_{i,j}(s) \\ &+ 2B_{ijpqr}u_{r,pq}(s)u_{i,j}(s) + C_{ijkmnr}u_{r,mn}(s)u_{k,ij}(s) \\ &+ \alpha k_{ij}\theta_{,i}(t)\theta_{,j}(t) + d\theta^2(t) + \alpha h\dot{\theta}^2(t) \\ &+ 2h\theta(t)\dot{\theta}(t)]\, dV + \int_0^t \int_B [k_{ij}\theta_{,i}(s)\theta_{,j}(s) \\ &+ (\alpha d - h)\dot{\theta}^2(s)]\, dV\, ds = E(0), \quad t \in [0,\infty) \end{aligned} \tag{25}$$

PROOF. Using the equation of motion (14) and some basic rules of derivation, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}[\rho\dot{u}_i(s)\dot{u}_i(s)] &= \rho\dot{u}_i(s)\ddot{u}_i(s) \\ &= \dot{u}_i(s)[t_{ij,j}(s) + \mu_{kji,kj}(s)] \\ &= [t_{ij}(s)\dot{u}_i(s)]_{,j} + [\mu_{kji}(s)\dot{u}_i(s)]_{,kj} \\ &\quad - t_{ij}(s)\dot{u}_{i,j}(s) - \mu_{kji}(s)\dot{u}_{i,kj}(s) \end{aligned} \tag{26}$$

If we take into considerations the constitutive Eq. (3), the relation (26) acquires the form

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}[\rho\dot{u}_i(s)\dot{u}_i(s)] \\ &\quad = [t_{ij}(s)\dot{u}_i(s)]_{,j} + [\mu_{kji}(s)\dot{u}_i(s)]_{,kj} - A_{ijrs}u_{r,}(s)\dot{u}_{i,j}(s) \\ &\qquad - B_{ijpqr}[u_{r,pq}(s)\dot{u}_{i,j}(s) + \dot{u}_{r,pq}(s)u_{i,j}(s)] \\ &\qquad - C_{ijkmnr}u_{r,mn}(s)\dot{u}_{k,ij}(s) \\ &\qquad - [a_{ij}\dot{u}_{i,j}(s) + c_{ijk}\dot{u}_{k,ij}(s)][\theta(s) + \alpha\dot{\theta}(s)] \end{aligned}$$

which can be restated in the form

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}&[\rho\dot{u}_i(s)\dot{u}_i(s)+A_{ijrs}u_{r,s}(s)u_{i,j}(s)\\&\quad+2B_{ijpqr}u_{r,pq}(s)u_{i,j}(s)+C_{ijkmnr}u_{r,mn}(s)u_{k,ij}(s)]\\&=[t_{ij}(s)\dot{u}_i(s)]_{,j}+[\mu_{kji}(s)\dot{u}_i(s)]_{,kj}\\&\quad-[a_{ij}\dot{u}_{i,j}(s)+c_{ijk}\dot{u}_{k,ij}(s)][\theta(s)+\alpha\dot{\theta}(s)]\end{aligned}\tag{27}$$

The last row of the relation (27) is now transformed by using Eq. (15). Then we use the constitutive equation for heat flux $q_i$, thus we get to

$$\begin{aligned}&-[a_{ij}\dot{u}_{i,j}(s)+c_{ijk}\dot{u}_{k,ij}(s)][\theta(s)+\alpha\dot{\theta}(s)]\\&\quad=k_{ij}\theta_{,ij}(s)[\theta(s)+\alpha\dot{\theta}(s)]\\&\qquad-[\dot{\theta}(s)+h\ddot{\theta}(s)][d\theta(s)+\alpha\dot{\theta}(s)]\end{aligned}\tag{28}$$

Substituting (28) in (27) so that and after some simple calculations, relation (27) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}&[\rho\dot{u}_i(s)\dot{u}_i(s)+A_{ijrs}u_{r,s}(s)u_{i,j}(s)\\&\quad+2B_{ijpqr}u_{r,pq}(s)u_{i,j}(s)+C_{ijkmnr}u_{r,mn}(s)u_{k,ij}(s)\\&\quad+\alpha k_{ij}\theta_{,i}(s)\theta_{,j}(s)+d\theta^2(s)+\alpha h\dot{\theta}^2(s)+2h\theta(s)\dot{\theta}(s)]\\&\quad+k_{ij}\theta_{,i}(s)\theta_{,j}(s)+(\alpha d-h)\dot{\theta}^2(s)\\&=[t_{ij}(s)\dot{u}_i(s)]_{,j}+[\mu_{kji}(s)\dot{u}_i(s)]_{,kj}\end{aligned}\tag{29}$$

At last we integrate equality (29) over $B\times(0,t)$ and by using the divergence theorem, the boundary conditions (16) and the initial conditions (13) we arrive at the desired result (25) and the proof of Theorem 1 is completed.

Theorem 2. *Consider $((u_i),\theta)$ a solution of the mixed initial boundary value problem defined by Eqs.* (14) *and* (15), *the boundary conditions* (10) *and initial conditions* (13). *Supposing that*

$$(u_i^0)\in\mathbf{W}_1(B),\quad (u_i^1)\in\mathbf{W}_0(B),\quad \theta^0\in W_1(B),\quad \theta^1\in W_0(B)$$

we obtain the following identity

$$\begin{aligned}&2\int_B\rho u_i(t)\dot{u}_i(t)\,dV+\int_B\left\{(\alpha d-h)\theta^2(t)\right.\\&\quad\left.+k_{ij}\left[\int_0^t\theta_{,i}(\xi)\,d\xi\right]\left[\int_0^t\theta_{,j}(\xi)\,d\xi\right]\right\}dV\\&\quad+2\int_B\alpha k_{ij}\theta_{,i}(t)\left[\int_0^t\theta_{,j}(\xi)\,d\xi\right]dV\\&=2\int_0^t\int_B[A\rho\dot{u}_i(s)\dot{u}_i(s)-{}_{ijrs}u_{r,s}(s)u_{i,j}(s)\\&\quad-2B_{ijpqr}u_{r,pq}(s)u_{i,j}(s)-C_{ijkmnr}u_{r,mn}(s)u_{k,ij}(s)\\&\quad-d\theta^2(s)-\alpha h\dot{\theta}^2(s)-2h\theta(s)\dot{\theta}(s)\\&\quad+\alpha k_{ij}\theta_{,i}(s)\theta_{,j}(s)]\,dV\,ds\\&\quad+2\int_B[\rho u_i^0u_i^1+(\alpha d-h)(\theta^0)^2]\,dV\\&\quad-2\int_0^t\int_B(a-\rho\eta^0)[\theta(s)+\alpha\dot{\theta}(s)]\,dV\,ds\end{aligned}\tag{30}$$

*for $t\in[0,\infty)$. Here $\rho\eta^0$ has the expression*

$$\rho\eta^0=a+d\theta^0+h\theta^1-b_i\theta^0_{,i}-a_{ij}u^0_{i,j}-c_{ijk}u^0_{k,ij}\tag{31}$$

Proof. With the help of the Eq. (14), the symmetry relations (5) and the constitutive relations (3) we get

$$\begin{aligned}\frac{d}{ds}&[\rho u_i(s)\dot{u}_i(s)]\\&=\rho\dot{u}_i(s)\dot{u}_i(s)+\rho u_i(s)\ddot{u}_i(s)=\rho\dot{u}_i(s)\dot{u}_i(s)\\&\quad+[u_i(s)t_{ij}(s)]_{,j}+[u_i(s)\mu_{kij}(s)]_{,ij}-A_{ijrs}u_{r,s}(s)u_{i,j}(s)\\&\quad-2B_{ijpqr}u_{r,pq}(s)u_{i,j}(s)-C_{ijkmnr}u_{r,mn}(s)u_{k,ij}(s)\\&\quad-a_{ij}[\theta(s)+\alpha\dot{\theta}(s)]u_{i,j}(s)\\&\quad-c_{kij}[\theta(s)+\alpha\dot{\theta}(s)]u_{k,ij}(s)\end{aligned}\tag{32}$$

With the intention to get another form for the expression of the last row of relationship (32), we integrate with respect to the time variable in the energy Eq. (15) and then we use the initial conditions (13) so that we deduce

$$\begin{aligned}&a_{ij}u_{i,j}(s)+c_{ijk}u_{k,ij}(s)\\&\quad=d\theta(s)+h\dot{\theta}(s)-\left[k_{ij}\int_0^s\theta_{,j}(\xi)\,d\xi\right]_{,i}+a-\rho\eta^0\end{aligned}\tag{33}$$

In (33) we multiply by $[\theta(s)+\alpha\dot{\theta}(s)]$ and the resulting expression will be replaced in (32). Thus, we find the equality

$$\begin{aligned}\frac{d}{ds}(\rho u_i\dot{u}_i)&=\rho\dot{u}_i(s)\dot{u}_i(s)+[u_i(s)t_{ij}(s)]_{,j}+[u_i(s)\mu_{kij}(s)]_{,ij}\\&\quad-A_{ijrs}u_{r,s}(s)u_{i,j}(s)-2B_{ijpqr}u_{r,pq}(s)u_{i,j}(s)\\&\quad-C_{ijkmnr}u_{r,mn}(s)u_{k,ij}(s)-d\theta^2(s)-\alpha h\dot{\theta}^2(s)\\&\quad-2h\theta(s)\dot{\theta}(s)-(\alpha d-h)\theta(s)\dot{\theta}(s)\\&\quad+\alpha k_{ij}\theta_{,i}(s)\theta_{,j}(s)\\&\quad+\left\{k_{ij}\left[\int_0^s\theta_{,j}(\xi)\,d\xi\right][\theta(s)+\alpha\dot{\theta}(s)]\right\}_{,i}\\&\quad-(a-\rho\eta^0)\left[\theta(s)+\alpha\dot{\theta}(s)\right]\\&\quad-\alpha k_{ij}[\dot{\theta}_{,i}(s)\int_0^s\theta_{,j}(\xi)\,d\xi+\theta_{,i}(s)\theta_{,j}(s)]\\&\quad-k_{ij}\theta_{,i}(s)\int_0^s\theta_{,j}(\xi)\,d\xi\end{aligned}\tag{34}$$

At last, we integrate the identity (34) over $B\times(0,t)$, then apply the divergence theorem and finally, we employ the boundary conditions (16), the initial conditions (13) and the symmetry relations (5). In this way we obtain the desired result (30). Thus, the proof of Theorem 2 is concluded.

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THEOREM 3. *Consider* $((u_i), \theta)$ *a solution of the mixed initial boundary value problem defined by* (14), (15), (16) *and* (13). *Whether the conditions*

$$(u_i^0) \in \mathbf{W}_1(B), \quad (u_i^1) \in \mathbf{W}_0(B), \quad \theta^0 \in W_1(B), \quad \theta^1 \in W_0(B)$$

*then we have the following identity*

$$\begin{aligned}
&2\int_B \rho u_i(t)\dot{u}_i(t)\,dV + \int_B \Big\{(\alpha d - h)\theta^2(t) \\
&\quad + k_{ij}\Big[\int_0^s \theta_{,i}(\xi)\,d\xi\Big]\Big[\int_0^s \theta_{,j}(\xi)\,d\xi\Big]\Big\}\,dV \\
&\quad + 2\int_B \alpha k_{ij}\theta_{,i}(t)\Big[\int_0^s \theta_{,j}(\xi)\,d\xi\Big]\,dV \\
&= \int_B \rho[u_i^1 u_i(2t) + u_i^0 \dot{u}_i(2t)]\,dV \\
&\quad + \int_B \Big\{(\alpha d - h)\theta^0\theta(2t) \\
&\qquad + \alpha k_{ij}\theta^0_{,i}\Big[\int_0^{2t} \theta_{,j}(\xi)\,d\xi\Big]\Big\}\,dV \\
&\quad + \int_0^t \int_B (a - \rho\eta^0)\{\theta(t+s) - \theta(t-s) \\
&\quad + \alpha[\dot{\theta}(t+s) - \dot{\theta}(t-s)]\}\,dV\,ds
\end{aligned} \tag{35}$$

for $t \in [0, \infty)$. *Here* $\eta^0$ *is expressed as given in* (31).

PROOF. It is not difficult to prove the following identity

$$\frac{d}{ds}\{\rho[f_i(s)\dot{g}_i(s) - \dot{f}_i(s)g_i(s)]\} = \rho[f_i(s)\ddot{g}_i(s) - \ddot{f}_i(s)g_i(s)]$$

where $f_i(x, s)$ and $g_i(x, s)$ are twice continuously differentiable functions with respect to the time variable $s$. By integrating the above identity over $B \times (0, t)$ one obtains

$$\begin{aligned}
&\int_B \rho[f_i(s)\dot{g}_i(s) - \dot{f}_i(s)g_i(s)]\,dV \\
&\quad = \int_0^t \int_B \rho[f_i(s)\ddot{g}_i(s) - \ddot{f}_i(s)g_i(s)]\,dV\,ds \\
&\qquad + \int_B \rho[f_i(0)\dot{g}_i(0) - \dot{f}_i(0)g_i(0)]\,dV
\end{aligned} \tag{36}$$

If we substitute in (36) the functions $f_i$ and $g_i$ as follows

$$f_i(x, s) = u_i(x, t-\tau), \quad g_i(x, s) = u_i(x, t+\tau)$$
$$\tau[0, t], \quad t \in (0, \infty)$$

then we lead to the following identity

$$\begin{aligned}
&2\int_B \rho u_i(t)\dot{u}_i(t)\,dV \\
&\quad = \int_B \rho[u_i^0\dot{u}_i(2t) + u_i^1 u_i(2t)]\,dV \\
&\qquad + \int_0^t \int_B \rho[u_i(t+s)\ddot{u}_i(t-s) \\
&\qquad - u_i(t-s)\ddot{u}_i(t+s)]\,dV\,ds, \quad t \in [0, \infty)
\end{aligned} \tag{37}$$

Inertial terms that appear in the last integral of (37) will be eliminated in the following. Taking into account the equations of motion (14) and the symmetry relations (5) one obtains

$$\begin{aligned}
&\rho[u_i(t+s)\ddot{u}_i(t-s) - u_i(t-s)\ddot{u}_i(t+s)] \\
&\quad = [u_i(t+s)t_{ji}(t-s) - u_i(t-s)t_{ji}(t+s)]_{,j} \\
&\qquad + [u_i(t+s)\mu_{ijk}(t-s) - u_i(t-s)\mu_{ijk}(t+s)]_{,ij} \\
&\qquad + a_{ij}[\theta(t+s) + \alpha\dot{\theta}(t+s)]u_{i,j}(t-s) \\
&\qquad - a_{ij}[\theta(t-s) + \alpha\dot{\theta}(t-s)]u_{i,j}(t+s) \\
&\qquad + c_{ijk}[\theta(t+s) + \alpha\dot{\theta}(t+s)]u_{k,ij}(t-s) - c_{ijk}[\theta(t-s) \\
&\qquad + \alpha\dot{\theta}(t-s)]u_{k,ij}(t+s)
\end{aligned} \tag{38}$$

On the other hand, with the help of the equation of energy (15) and using the same idea as in the proof of relation (33) one obtains

$$\begin{aligned}
&a_{ij}[\theta(t+s) + \alpha\dot{\theta}(t+s)]u_{i,j}(t-s) \\
&\quad - a_{ij}[\theta(t-s) + \alpha\dot{\theta}(t-s)]u_{i,j}(t+s) + c_{ijk}[\theta(t+s) \\
&\quad + \alpha\dot{\theta}(t+s)]u_{k,ij}(t-s) - c_{ijk}[\theta(t-s) \\
&\quad + \alpha\dot{\theta}(t-s)]u_{k,ij}(t+s) \\
&= (a - \rho\eta^0)\Big\{\theta(t-s) - \theta(t+s) + \alpha[\dot{\theta}(t-s) - \dot{\theta}(t+s)]\Big\} \\
&\quad + (\alpha d - h)[\theta(t-s)\dot{\theta}(t+s) - \theta(t+s)\dot{\theta}(t-s)] \\
&\quad + k_{ij}\Big\{\theta_{,i}(t+s)\Big[\int_0^{t-s}\theta_{,j}(\xi)\,d\xi\Big] \\
&\qquad - \theta_{,i}(t-s)\Big[\int_0^{t-s}\theta_{,j}(\xi)\,d\xi\Big]\Big\} \\
&\quad + \alpha k_{ij}\Big\{\dot{\theta}_{,i}(t+s)\Big[\int_0^{t-s}\theta_{,j}(\xi)\,d\xi\Big] - \theta_{,i}(t-s)\theta_{,j}(t+s)\Big\} \\
&\quad + \alpha k_{ij}\Big\{\dot{\theta}_{,i}(t-s)\Big[\int_0^{t+s}\theta_{,j}(\xi)\,d\xi\Big] - \theta_{,i}(t+s)\theta_{,j}(t-s)\Big\} \\
&\quad + \Big\{k_{ij}[\theta(t-s) + \alpha\dot{\theta}(t-s)]\int_0^{t+s}\theta_{,j}(\xi)\,d\xi\Big\}_{,i} \\
&\quad - \Big\{k_{ij}[\theta(t+s) + \alpha\dot{\theta}(t+s)]\int_0^{t-s}\theta_{,j}(\xi)\,d\xi\Big\}_{,i}
\end{aligned} \tag{39}$$

At last, we substitute (39) into (38), integrate over $B \times [0, t]$ and then we will use the divergence theorem and the boundary conditions (16). Thus we are lead to the following identity

$$\begin{aligned}
&2\int_B \rho u_i(t)\dot{u}_i(t)\,dV \\
&\quad = \int_B \rho[u_i^0\dot{u}_i(2t) + u_i^1 u_i(2t)]\,dV + \int_0^t \int_B (a - \rho\eta^0)\{\theta(t+s) \\
&\qquad - \theta(t-s) + \alpha[\dot{\theta}(t+s) - \dot{\theta}(t-s)]\}\,dV\,ds
\end{aligned}$$

$$+\int_0^t \int_B \left\{ (\alpha d - h)\frac{d}{ds}[\theta(t+s)\theta(t-s)] + \frac{d}{ds}\left[ k_{ij}\int_0^{t+s} \theta_{,i}(\xi)\, d\xi \int_0^{t-s} \theta_{,j}(\xi)\, d\xi \right] \right\} dV\, ds + \int_0^t \int_B [\alpha k_{ij}\theta_{,i}(t+s) \int_0^{t-s} \theta_{,j}(\xi)\, d\xi + \alpha k_{ij}\theta_{,i}(t-s)\int_0^{t+s} \theta_{,j}(\xi)\, d\xi]\, dV\, ds \tag{40}$$

Now, by using the initial conditions (13) into (40), one obtains the desired result (35) such that Theorem 3 is proved.

## 4. EQUIPARTITION OF ENERGY

Based on identities (25), (30) and (35) and using the assumptions made in Section 2, we can now prove the asymptotic partition for non-simple thermoelastic bodies.

We begin with the introduction of Cesaro means of all components of energy contained in the identity (25), namely, kinetic energy, strain energy, potential energy, thermal energy, energy of diffusion and energy of dissipation, as follows

$$\begin{aligned}
K(t) &\equiv \frac{1}{2t}\int_0^t \int_B \rho \dot{u}_i(s)\dot{u}_i(s)\, dV\, ds \\
S(t) &\equiv \frac{1}{2t}\int_0^t \int_B [A_{ijrs}u_{r,s}(s)u_{i,j}(s) + 2B_{ijpqr}u_{r,pq}(s)u_{i,j}(s) + C_{ijkmnp}u_{p,mn}(s)u_{k,ij}(s)]\, dV\, ds \\
P(t) &\equiv \frac{1}{2t}\int_0^t \int_B \alpha k_{ij}\theta_{,i}(s)\theta_{,j}(s)\, dV\, ds \\
T(t) &\equiv \frac{1}{2t}\int_0^t \int_B d\theta^2(s)\, dV\, ds \\
F(t) &\equiv \frac{1}{2t}\int_0^t \int_B \alpha h \dot{\theta}^2(s)\, dV\, ds \\
D(t) &\equiv \frac{1}{t}\int_0^t \int_0^t \int_B [k_{ij}\theta_{,i}(\xi)\theta_{,j}(\xi) + (\alpha d - h)\dot{\theta}^2(\xi)]\, dV\, d\xi\, ds
\end{aligned} \tag{41}$$

The main result of our study is formulated and proved in the following theorem.

THEOREM 4. *Consider* $((u_i), \theta)$ *a solution of the mixed initial boundary value problem for non-simple thermoelastic bodies defined by* (14), (15), (16) *and* (13). *We assume that the hypotheses from Section 2 are satisfied. Then, for any initial data*

$$(u_i^0) \in \mathbf{W}_1(B), \quad (u_i^1) \in \mathbf{W}_0(B), \quad \theta^0 \in W_1(B), \quad \theta^1 \in W_0(B)$$

we have

$$\lim_{t\to\infty} P(t) = 0, \quad \lim_{t\to\infty} F(t) = 0 \tag{42}$$

Also, we have the following relations

(i) If *meas* $(\partial B_2) \neq 0$, then

$$\lim_{t\to\infty} T(t) = 0 \tag{43}$$

(ii) If *meas* $(\partial B_2) = 0$, then

$$\lim_{t\to\infty} T(t) = \frac{1}{2}\int_B \frac{1}{d}(d\theta^* + h\dot{\theta}^*)\, dV \tag{44}$$

(iii) If *meas* $(\partial B_1) \neq 0$, then

$$\lim_{t\to\infty} K(t) = \lim_{t\to\infty} S(t) \tag{45}$$

$$\lim_{t\to\infty} D(t) = E(0) - 2\lim_{t\to\infty} K(t) = E(0) - 2\lim_{t\to\infty} S(t) \tag{46}$$

(iv) If *meas* $(\partial B_1) = 0$, then

$$\lim_{t\to\infty} K(t) = \lim_{t\to\infty} S(t) + \frac{1}{2}\int_B \rho \dot{u}_i^* \dot{u}_i^*\, dV \tag{47}$$

$$\lim_{t\to\infty} D(t) = E(0) - 2\lim_{t\to\infty} K(t) + \frac{1}{2}\int_B \rho \dot{u}_i^* \dot{u}_i^*\, dV = E(0) - 2\lim_{t\to\infty} S(t) - \frac{1}{2}\int_B \rho \dot{u}_i^* \dot{u}_i^*\, dV \tag{48}$$

PROOF. Let's start a demonstration addressing relation (42). Based on the hypotheses of Section 2, we will use the conservation law of the energy (25). So, by hypotheses (10) one obtains

$$\begin{aligned}
&d\theta^2(t) + \alpha h \dot{\theta}^2(t) + 2h\theta(t)\dot{\theta}(t) \\
&= \frac{1}{d}[d\theta(t) + h\dot{\theta}(t)]^2 + \frac{h}{d}(\alpha d - h)\dot{\theta}^2(t) \\
&= \frac{h}{\alpha}[\theta(t) + \alpha\dot{\theta}(t)]^2 + \frac{1}{\alpha}(d\alpha - h)\theta^2(t) \geq 0
\end{aligned} \tag{49}$$

By using the relations (41) and (25), we deduce

$$F(t) \leq \frac{1}{2t}\frac{h\alpha}{\alpha d - h}E(0) \tag{50}$$

$$P(t) \leq \frac{\alpha}{2t}E(0) \tag{51}$$

Using the criterion of increasing, by passing to the limit in (50) and (51), we obtain immediately the relation (42). (i) Suppose that *meas* $(\partial B_1) \neq 0$. It is easy to prove that $\theta \in \hat{W}_1(B)$. Therefore, we can apply the Poincare's inequality (22) such that from the identity (25) one obtains

$$\int_0^t \int_B d\theta^2(s)\, dV\, ds \leq \frac{d}{m_2}\int_0^t \int_B k_{ij}\theta_{,i}(s)\theta_{,j}(s)\, dV\, ds \leq \frac{d}{m_2}E(0) \tag{52}$$

Taking into account the definition of thermal energy in (41), from the relation (52) we obtain the conclusion (43).

(ii) Let us suppose that $meas\ (\partial B_2)=0$. Thus, we can use the decomposition (24) and the fact that $\chi \in \hat{W}_1(B)$ in order to obtain the following identity

$$\begin{aligned}\int_B \theta^2(s)\,dV &= \int_B \left[\theta^* + \frac{h}{d}\dot{\theta}^*\right]^2 dV + \int_B \chi^2(s)\,dV \\ &\quad - 2\int_B \frac{h}{d}\left[\theta^* + \frac{h}{d}\dot{\theta}^*\right]\theta^* \exp\left(-\frac{d}{h}t\right) dV \\ &\quad + \frac{h^2}{d^2}(\dot{\theta}^*)^2 \exp\left(-\frac{2d}{h}t\right) dV \end{aligned} \tag{53}$$

From the relations (41) and (53) we deduce

$$\begin{aligned} T(t) &= \frac{1}{2}\int_B \frac{1}{d}(d\theta^* + \dot{\theta}^*)^2\,dV + \frac{1}{2t}\int_0^t\int_B d\chi^2(s)\,dV\,ds \\ &\quad - \frac{1}{t}\left[1-\exp\left(-\frac{d}{h}t\right)\right]\int_B \frac{h^2}{d^2}\theta^*(d\theta^*+\dot{\theta}^*)\,dV \\ &\quad + \frac{1}{4t}\left[1-\exp\left(-\frac{2d}{h}t\right)\right]\int_B \frac{h^3}{d^2}(\dot{\theta}^*)^2\,dV \end{aligned} \tag{54}$$

By using the Poincare's inequality (22), the identity (25) and the fact that $\chi \in \hat{W}_1(B)$ one obtains

$$\begin{aligned} \frac{1}{2t}\int_0^t\int_B d\chi^2(s)\,dV\,ds &\le \frac{d}{2tm_2}\int_0^t\int_B k_{ij}\chi_{,i}(s)\chi_{,j}(s)\,dV\,ds \\ &= \frac{d}{2tm_2}\int_0^t\int_B k_{ij}\theta_{,i}(s)\theta_{,j}(s)\,dV\,ds \le \frac{d}{2tm_2}E(0) \end{aligned} \tag{55}$$

Passing to the limit in (54), as $t\to\infty$ and taking into account (10) and (55), we arrive to (44).

We now use the relation (49), the energy conservation law (25) and the hypotheses of Section 2 in order to obtain the following estimates:

$$\int_B \theta^2(t)\,dV \le 2\alpha\frac{1}{\alpha d - h}E(0),\quad t\in[0,\infty) \tag{56}$$

$$\int_B \rho\dot{u}_i(t)\dot{u}_i(t)dV \le 2E(0),\quad t\in[0,\infty) \tag{57}$$

$$\int_0^t\int_B k_{ij}\chi_{,i}(s)\chi_{,j}(s)\,dV\,ds \le E(0),\quad t\in[0,\infty) \tag{58}$$

$$\int_B \dot{\theta}^2(s)\,dV\,ds \le \frac{1}{\alpha d-h}E(0),\quad t\in[0,\infty) \tag{59}$$

On the other hand, the identities (30) and (35) imply

$$\begin{aligned} &\frac{1}{2t}\int_0^t\int_B [\rho\dot{u}_i(s)\dot{u}_i(s) - A_{ijrs}u_{i,j}(s)u_{m,n}(s) \\ &\quad - 2B_{ijpqr}u_{i,j}(s)u_{p,qr}(s) - C_{ijkmnr}u_{k,ij}(s)u_{r,mn}(s)]\,dV \\ &= \frac{1}{4t}\int_B \rho[u_i^1u_i(2t) + u_i^0\dot{u}_i(2t)]\,dV \\ &\quad + \frac{1}{2t}\int_0^t\int_B [d\theta^2(s) + 2h\theta(s)\dot{\theta}(s) + \alpha h\dot{\theta}^2(s) \\ &\quad - \alpha k_{ij}\theta_{,i}(s)\theta_{,j}(s)]\,dV\,ds - \frac{1}{4t}\int_B (\alpha d - h)(\theta^0)^2\,dV \\ &\quad + \frac{1}{2t}\int_0^t\int_B (a-\rho\eta^0)[\theta(s)+\alpha\dot{\theta}(s)]\,dV\,ds \\ &\quad - \frac{1}{2t}\int_B \rho u_i^0u_i^1\,dV + \frac{1}{4t}\int_B\Big[(\alpha d-h)\theta^0(2t) \\ &\quad + \alpha k_{ij}\theta^0_{,i}\int_0^{2t}\theta_{,j}(\xi)\,d\xi\Big]dV \\ &\quad + \frac{1}{4t}\int_0^t\int_B (a-\rho\eta^0)\{\theta(t+s)-\theta(t-s) \\ &\quad + \alpha\frac{d}{ds}[\theta(t+s)+\theta(t-s)]\}\,dV\,ds \end{aligned} \tag{60}$$

$$\begin{aligned} &K(t)-L(t) \\ &= \frac{1}{4t}\int_B \{(\alpha d-h)[\theta^0 + \alpha(a-\rho\eta^0)]\}[\theta(2t)-\theta^0]\,dV \\ &\quad + \frac{1}{4t}\int_0^{2t}\int_B \alpha k_{ij}\theta^0_{,i}\theta_{,j}(s)\,dV\,ds \\ &\quad + \frac{1}{t}\int_0^t\int_B h\theta(s)\dot{\theta}(s)\,dV\,ds - \frac{1}{2t}\int_B \rho u_i^0u_i^1\,dV \\ &\quad + F(t) - P(t) + \frac{1}{4t}\int_B \rho u_i^0\dot{u}_i(2t)\,dV \\ &\quad + \frac{1}{4t}\int_B \rho u_i^1u_i(2t)\,dV + T(t) \\ &\quad + \frac{1}{4t}\int_0^t\int_B (a-\rho\eta^0)[\theta(t+s)+\theta(s)]\,dV\,ds \end{aligned} \tag{61}$$

Now we will use the Schwarz and Cauchy inequalities on the right side of (61). Then, by using the relations (50)–(52), (56)–(58) we get

$$\begin{aligned} &\left|-\frac{1}{2t}\int_B \rho u_i^0u_i^1\,dV\right| \le \frac{1}{4t}\int_B \rho[u_i^0u_i^0 + u_i^1u_i^1]\,dV \\ &\left|\frac{1}{4t}\int_B \{(\alpha d-h)[\theta^0+\alpha(a-\rho\eta^0)]\}[\theta(2t)-\theta^0]\,dV\right| \\ &\le \frac{\alpha}{2t(\alpha d-h)}E(0) \\ &\quad + \frac{1}{8t}\int_B \{[(\alpha d-h)[\theta^0+\alpha(a-\rho\eta^0)]]^2 + 2(\theta^0)^2\}\,dV \\ &\left|\frac{1}{4t}\int_0^{2t}\int_B \alpha k_{ij}\theta^0_{,i}\theta_{,j}(s)\,dV\,ds\right| \\ &\quad \le \frac{1}{4t}\left(\int_0^{2t}\int_B \alpha k_{ij}\theta^0_{,i}\theta_{,j}(s)\,dV\,ds\right)^{1/2} \\ &\quad \times \left(\int_0^{2t}\int_B \alpha k_{ij}\theta_{,i}(s)\theta_{,j}(s)\,dV\,ds\right)^{1/2} \\ &\quad \le \frac{1}{2}\left(\frac{\alpha}{2t}E(0)\int_B \alpha k_{ij}\theta^0_{,i}\theta^0_{,j}\,dV\right)^{1/2} \end{aligned} \tag{62}$$

$$\left|\frac{1}{t}\int_0^t h\theta(s)\dot{\theta}(s)\,dV\,ds\right| \le \frac{1}{t}\left(\int_0^t\int_B h\theta^2(s)\,dV\,ds\right)^{1/2} \times \left(\int_0^t\int_B h\dot{\theta}^2(s)\,dV\,ds\right)^{1/2} \le \sqrt{\frac{2\alpha}{t}}\frac{h}{\alpha d-h}E(0)$$



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$$\left|\frac{1}{4t}\int_B \rho u_i^0 \dot{u}_i(2t)\,dV\right| \le \frac{1}{8t}\int_B \rho u_i^0 u_i^0\,dV + \frac{1}{4t}E(0)$$

(iii) Assume that $meas\ (\partial B_1) \neq 0$. Since $(u_i) \in W_1(B)$, from (6), (21) and (25) one obtains

$$\begin{aligned}\int_B \rho u_i(s) u_i(s)\,dV &\le \frac{k}{m_1}\int_B [A_{ijrs} u_{i,j}(s) u_{r,s}(s)\\ &\quad + 2B_{ijpqr} u_{i,j}(s) u_{p,qr}(s) + C_{ijkpqr} u_{k,ij}(s) u_{r,pq}(s)]\,dV\\ &\le \frac{2k}{m_1}E(0), \quad s\in[0,\infty) \end{aligned} \tag{63}$$

such that we deduce

$$\left|\frac{1}{4t}\int_B \rho u_i^1 u_i(2t)\,dV\right| \le \frac{1}{8t}\int_B \rho u_i^1 u_i^1\,dV + \frac{k}{4tm_1}E(0) \tag{64}$$

If we suppose that $meas\ (\partial B_1) \neq 0$, then we have

$$\begin{aligned}&\left|T(t) + \frac{1}{4t}\int_0^t\int_B (a-\rho\eta^0)[\theta(t+s)+\theta(s)]\,dV\,ds\right|\\ &\quad\le T(t) + \frac{1}{4t}\left(\int_0^t\int_B (a-\rho\eta^0)^2\,dV\,ds\right)^{1/2}\\ &\quad\quad\times\left(\int_0^t\int_B [\theta(t+s)+\theta(s)]^2\,dV\,ds\right)^{1/2} \le T(t)\\ &\quad\quad+\left(\frac{1}{2d}\int_0^t\int_B (a-\rho\eta^0)^2\,dV\,ds\right)^{1/2}[T(2t)]^{1/2} \end{aligned} \tag{65}$$

By passing to the limit in (61) as $t\to\infty$ and taking into account, the estimates (62), (64), (65) and the relations (42) and (43) we conclude that the relation (45) holds.

Now, we suppose that $meas\ (\partial B_2) = 0$. We will use the decompositions from (19) and (24) and taking into account the relations (20), (31) and the expression of $\eta^0$ from (54), we conclude that the following identity holds

$$\begin{aligned}&T_C(t) + \frac{1}{4t}\int_0^t\int_B (a-\rho\eta^0)[\theta(t+s)+\theta(s)]\,dV\,ds\\ &= -\frac{1}{4t}\int_B \frac{h^2}{d^2}\dot{\theta}^*\left[d\theta^* + \frac{3}{2}h\dot{\theta}^*\right]\left[\exp\left(-\frac{2d}{h}t\right)-1\right]dV\\ &\quad + \frac{1}{2t}\int_0^t\int_B d\chi^2(s)\,dV\,ds + \frac{1}{4t}\int_B \frac{h^2}{d^2}\dot{\theta}^*(d\theta^* + \alpha\dot{\theta}^*)\\ &\quad\times\left[\exp\left(-\frac{d}{h}t\right)-1\right]dV\\ &\quad + \frac{1}{4t}\int_0^t\int_B [a_{ij}u^0_{i,j} + c_{ijk}u^0_{k,ij} + b_i\theta^0_{,i} - dT^0 - hT^1]\\ &\quad\times[\chi(t+s)+\chi(s)]\,dV\,ds \end{aligned} \tag{66}$$

Now, we shall use the Schwarz and Cauchy inequalities in (66) and taking into account the relation (55) and we obtain

$$\lim_{t\to\infty}\left\{T_C(t) + \frac{1}{4t}\int_0^t\int_B (a-\rho\eta^0)[\theta(t+s)+\theta(s)]\,dV\,ds\right\} = 0 \tag{67}$$

By introducing the relations (42), (62), (64) and (67) into (61) we obtain again the conclusion (45). Also, it is no difficult to observe that the relation (4) is obtained from identity (25) by taking the Cesaro mean and by using the relations (42), (43) and (45).

(iv) In the case that $meas\ (\partial B_1) = 0$ we will use the decomposition (23), the decomposition (17) with conditions (18) and the fact that $(u_i) \in W_1(B)$ in order to obtain the following identity

$$\begin{aligned}\frac{1}{4t}\int_B \rho u_i^0 \dot{u}_i(2t)\,dV &= \frac{1}{4t}\int_B \rho u_i^* \dot{u}_i^* dV + \frac{1}{2}\int_B \rho \dot{u}_i^* \dot{u}_i^*\,dV\\ &\quad + \frac{1}{4t}\int_B \rho U_i^1 v_i(2t)\,dV \end{aligned} \tag{68}$$

Also, since $(v_i, \psi_i, \delta) \in W_1(B)$, the Korn's inequality (21) leads to the relation

$$\begin{aligned}&\frac{1}{4t}\int_B \rho v_i(s) v_i(s)\,dV \le \frac{k}{m_1}\int_B [A_{ijrs} v_{r,s}(s) v_{i,j}(s)\\ &\quad + 2B_{ijpqr} v_{i,j}(s) v_{r,pq}(s) + C_{ijkpqr} v_{k,ij}(s) v_{r,pq}(s)]\,dV\\ &= \frac{k}{m_1}\int_B [A_{ijrs} u_{i,j}(s) u_{r,s}(s) + 2B_{ijpqr} u_{i,j}(s) u_{r,pq}(s)\\ &\quad + C_{ijkpqr} u_{k,ij}(s) u_{r,pq}(s)]\,dV \le \frac{2k}{m_1}E(0)\\ &\quad s\in[0,\infty) \end{aligned} \tag{69}$$

Passing to the limit in (61), as $t\to\infty$, and taking into account the relations (42), (62), (65), (68) and (69) one obtains the conclusion (47).

At last, the relation (48) is obtained on the basis of identity (25) by taking the Cesaro mean and by using the relations (42), (43), (47) and (62). Thus, the proof of theorem 4 is complete.

## 5. CONCLUSION

If we take into account the relations (45) and (47), restricted to the class of initial data for which $\dot{u}_i^* = 0$, we obtain the asymptotic equipartition in mean of the kinetic and strain energies.

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